Amirkabir Journal of Civil Engineering

Amirkabir J. Civil Eng., 52(2) (2020) 93-98
DOI: 10.22060/ceej.2019.13840.5487

The Analyzing of the Discontinuity Problem by Enriched Interpolation Covers

H. Arzani* , E. Khoshbavar Rad
Department of Civil Engineering, Shahid Rajaee Teacher Training University, Tehran, Iran

ABSTRACT: The cover numerical method is based on unit partitioning, using enrichment functions of various orders, this method increases the accuracy at every point. In this method, unknowns are added to the points of not sufficient accuracy, regarding to the enrichment interpolating functions. The cover method has proved its efficiency in a variety of engineering problems. The Heaviside function models the displacement discontinuity on the crack boundary. Such functions are added to model the coefficient of stress intensity for the tip of the crack. In this paper, the proposed method has been verified by evaluating crack parameters for instances containing fixed fractures. In the end, three numerical instances containing central crack, edge crack and inclined crack with three different cracks are inspected. The comparison of the results from the presented method, with exact solutions and other solutions in the area of linear elasticity, proves the reliability and accuracy of the proposed method.

Review History:

Received: 2018-02-24
Revised: 2018-09-22
Accepted: 2018-09-29
Available Online: 2019-01-08

Keywords:

The interpolating cover Functions method
Discontinuous fields
Heaviside function
Crack

1. INTRODUCTION

The crack modeling is an important problem in forecasting the life of engineering structures, and over past decades it was paid much attention by many researchers. Many simulations using the finite element method with renewed mesh generation and mesh-free methods, were used to model crack growth as well as its extension. Extra computational cost is also encountered because of the need for fine elements in the crack area. When the crack growth enters the problem, the need for remeshing in these methods makes the simulation quite boring and time-consuming. Variables such as displacement, stress, and strain need to be interpreted in the new mesh which further increases the computational cost.

Calculation of stress intensity factor (SIF) plays a key role in the fracture mechanics. Among different kinds of numerical methods, the finite element method (FEM) is a representative. The FEM has already been used to solve a great number of crack problems. With the delimitations, for crack problems, the crack surface must be consistent with element edges, and the crack tip must be designed as a node. What is more, for accuracy reason, high mesh resolution is required around the crack tip even when singular elements are used. Considering the limitations of the FEM, recent years, several kinds of neo-numerical methods have been put forward to solve discontinuity problems with higher accuracy and efficiency in the framework of the partition of unity method (PUM) [1].

Arzani et al. obtain more accurate answers in discrete least-squares meshless method for solving problems in the field of fracture mechanics with the use of the CSS algorithm [2, 3].

The numerical manifold method (NMM) was first introduced by Shi. This is a method based on the combination of finite element and discrete displacement analysis (DDA) [4]. Shi et al., had proposed a numerical manifold method to analyze discontinuous deformations, which is an effective technique in solid mechanics field and has been the subject of many researches in recent years [5].

The choice of cover function is also important for proper NMM analysis, in particular when the size of each block is quite large. When the INMM is used for a crack problem, a cover is divided into two irregular sub-covers at the discontinuity. As a result, the method sometimes causes errors at the tip of a crack. To improve the precision of INMM, analytical solution near the tip of a crack is used in the present paper. The enrichment near the tip of a crack for linear elastic problems is achieved by the expansion of the basis functions with singular functions [6].

In recent years, Kim and Bathe [7] presented the method of enrichment method by the cover interpolation function, which was inspired by the manifold method. They used low order finite elements to mesh their model geometry to accelerate convergence. The method was based on enriched finite elements by the cover interpolation elements on each element. This method is also applicable to problems containing distorted elements, and the main idea is to increase the accuracy without excessive increase in degrees of freedom.

*Corresponding author's email: h.arzani@srttu.com

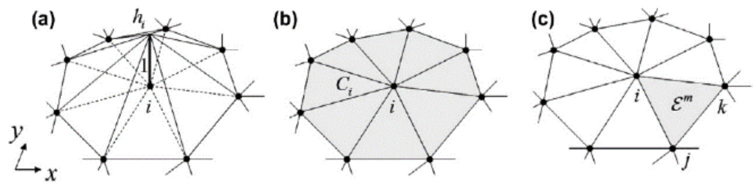

Fig. 1. interpolations: (a) usual linear nodal shape function, (b) cover region or elements affected by the interpolation cover, and (c) an element [7]

In their next movement [8], the researchers attempted to overcome these problems, but the enhancements made the method less general, and it was no longer applicable to entire elasticity problems. As an instance, a compact grid was necessary from the beginning to prevent an excessive increase in the order of interpolation function in all problems. Another important instance was the lack of existence of a comprehensive formulation for all problems by which the proper degree of interpolation functions are attained.

The authors, after inspecting the advantages and disadvantages of the interpolation functions method, have proposed a solution to automatically solve the problems in elasticity field, which has criterion and to evaluate standard error to elect proper order. The criterion was free of computational complexities associate with the tetragonal elements. In this paper, this method is generalized to be applied to discontinuous problems while still having its advantages.

General differential stress equations in elastostatic problem by requirements boundary conditions are solved in some benchmark examples, and a comparison between the proposed method and other available methods are presented.

2. DOMAIN DISCRETIZATION AND INTERPOLATION

In this step, the formulation of enriched finite elements using interpolating function is discussed briefly. If a standard discretization is used for meshing a domain, the accuracy of the response would depend on the size and size of the elements. In this type of enrichment, a covering subdomain for each node is considered. Every subdomain contains an interpolating function with the known degree. In these subdomains, higher-order functions are used concerning standard condition, which in turn leads to results with higher accuracies. In Figure 1a, function hi is a linear interpolating function for the node i. Its value is 1 at the node i , and in other nodes dependent to I is equal to 0. Elements attached to node i are considered subdomain of this node. The usage of linear functions for interpolating the subdomains results in approximations with a low amount of computations and a high rate. The region interpolated for the node I by the linear function of hi is dubbed the cover region of node I and is shown with Ci (Figure 1b). The cover region of triangular node M, with three nodes, i, j, k is the same as Figure 1c. It equals the axiom of cover areas c_i ، c_j and c_k. Bathe and Kim in [7] implied that the use of this method for tetragonal elements produces several problems.

After the determination of the cover zone of nodes, it is the turn of interpolating those cover zones. This is done by p-order polynomials. Interpolated value of unknown u at node i with regard to its cover zone is shown in Figure 6.

$$P_i^p[u] = u_i + \left[\bar{x}_i \quad \bar{y}_i \quad \bar{x}_i^2 \quad \bar{x}_i\bar{y}_i \quad \bar{y}_i^2 \quad \dots \quad \bar{y}_i^p\right] a_i \tag{5}$$

In relation 5, variables $(\bar{x}_i, \bar{y}_i)$ imply the distance from node I and vector u_i shows extra degrees of freedom related to node I in the cover zone i. Enriched approximation for the field variable u for an element is given in Equation 6:

$$u = \sum_{m=1}^{3} h_i u_i + H_i a_i \tag{6}$$

Where:

$$H_i = h_i\left[\bar{x}_i \quad \bar{y}_i \quad \bar{x}_i^2 \quad \bar{x}_i\bar{y}_i \quad \bar{y}_i^2 \quad \dots \quad \bar{y}_i^p\right] \tag{7}$$

With the summation of values in 7 over nodes in an element and combining Equations 6-8, after some algebra, its found that:

$$u = \sum_{m=1}^{3} h_i P_i^p \tag{8}$$

And thus, instead of the standard method of interpolating, the $h_i P_i^p$ method of interpolating has been used. Where P_i^p contains normal field values u_i plus the degrees of freedom related to cover functions. It is clear from the formulation that in addition to standard values of interpolating, higher orders of interpolating could be obtained by using enriched cover functions.

One advantage of mentioned is that it increases the interpolation accuracy only at those places where we haven't obtained the desired accuracy. In other regions, there is no need for enriched functions. It should be mentioned that determination of order, for enrichment functions is of much importance and if this order will be set to zero, no extra interpolation would take place, and standard interpolating takes place.

3. CRACK MODELING USING COVER INTERPOLATING FUNCTION METHOD

In the linear elastic fracture, the existence of crack causes displacement jump in the crack as well as singularity in stress at the tip of the crack. In this part, a cover interpolating method is proposed to solve discontinuity problems. For technical modeling of the crack tip in the cover interpolating method, enrichment functions are used as is in XFEM method [9].

In the next part, approximation in the regions containing discontinuity and regions close to crack tip are presented. A proper criterion is also introduced in an arbitrary grid to

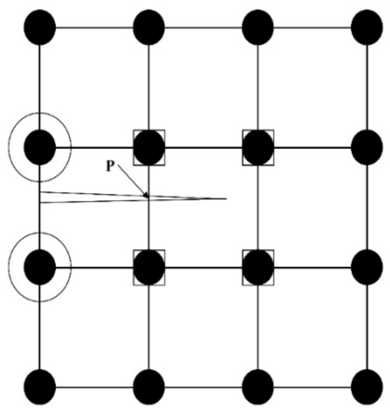

Fig. 3. Crack not aligned with a mesh, the circled nodes are enriched with the discontinuous function and the squared nodes with the tip enrichment functions. Enrichment with only the discontinuous function shortens the crack to point p

identify suitable points for enrichment, regarding the crack geometry. A brief explanation of stress coefficient computation is also discussed.

To model this discontinuity, H(x), which is a jump function or discontinuity function has been used which is defined in the local coordinate system as follows:

$$H(x,y)=\begin{cases}1 & for \;\; y>0 \\ -1 & for \;\; y<0\end{cases} \tag{9}$$

Where $H(x)=1$ at the upper edge of the crack and $H(x)=-1$ at the bottom of it. There is a convention in enrichment by jump function, which states enrichment is done on the points where the crack takes place. In a more general situation such as figure 3, the crack tip does not coincide with the element edge and the discontinuity may not be properly modeled by H(x) function. In this situation, jump enrichment on the circle's points suffices only for modeling of discontinuity up to point p.

As an instance, for discretization in Figure 3, the following approximation is used.

$$u^h=\sum_{i\in I}u_i\phi_i+\sum_{j\in J}b_j\phi_jH(x)+\sum_{k\in K}\phi_k\left(\sum_{l=1}^{4}c_k^lF_l(x)\right) \tag{10}$$

Where I is related to enrichment cover functions, J is related to a set of circular points and K is related to square points. $F_l(x)$ also is defined as follows.

$$\{F_l(r,\theta)\}=\left\{\sqrt{r}\sin\left(\frac{\theta}{2}\right),\sqrt{r}\cos\left(\frac{\theta}{2}\right),\ldots \right.$$
$$\left.\sqrt{r}\sin\left(\frac{\theta}{2}\right)\sin(\theta),\sqrt{r}\cos\left(\frac{\theta}{2}\right)\sin(\theta)\right\} \tag{11}$$

Where (r,θ) are the polar coordinates for the tip of the crack. It is noteworthy that the first part of the above relation $\sqrt{r}\sin(\theta/2)$ is related to the discontinuity on the crack surface while the other three parts are continues. In the present paper, low degree triangular elements are used to solve the problems. Above formulation has been generalized for an arbitrary crack in the triangular grid as follows:

$$u^h=\sum_{i\in I}u_i\phi_i+\sum_{j\in J}b_j\phi_jH(x)+$$
$$\sum_{k\in K_1}\phi_k\left(\sum_{l=1}^{3}c_k^{l1}F_l^1(x)\right)+\sum_{k\in K_2}\phi_k\left(\sum_{l=1}^{3}c_k^{l2}F_l^2(x)\right) \tag{12}$$

Where K1 and K2 are sets of points for the crack's first and second tips. The precise definition for these two sets is given in reference [10] as well as set J. Functions $F_l^1(x)$ and $F_l^2(x)$ are the same and regarding the local system, both crack tips (r_1,θ_1) and (r_2,θ_2) are defined as shown in Figure 4.

Table 1. Results of analysis of the central crack [10]

Method	KI	Error
Reference[11]	2.7641	-
XFEM [10]	2.7096	1.9
Interpolation cover-1	2.7201	1.5
Interpolation cover-2	2.7361	1.1
Interpolation cover-3	2.7411	1.01

4. NUMERICAL EXAMPLES

Three problems, central crack, inclined crack, and edge crack are considered and the stress coefficient is calculated for all three cases. The comparison between the exact solution and results obtained by the presented method proves the ability of this method in solving discontinuity problems.

4.1. Examples: central, Inclined central and edge crack subjected to tensional force

Consists of a plane with central crack subjected to tensional force at the upper edge, as shown in Figure 5. Problem's parameters consist of elasticity module $E=3\times10^7\ psi$, Poisson's ratio $\nu=0.25$, exerted stress $\sigma=1.0\ psi$, and related equations assume planar strain. The crack position is at $y=0.0$ and the x interval $x=-2.0-2.0$.

The geometry of the problem consists of the region $[-5.0,5.0]\times[-10.0,10.0]$. In this example, interpolating cover functions could be used to increase accuracy without the need to use finer grids. These functions could be used at nodes where the necessary accuracy is not obtained. In this investigation, interpolating cover functions of degree one, two and three are used.

Belytschko et al. in [10] have analyzed this problem with two separate grids using XFEM method. The first mesh contained 402 nodes and 25 enriched points at each crack tip while the second consisted of 1606 nodes and 64 points at each tip. The exact solution for this geometry is given in [11], where a limited correcting factor $K_I=2.7641$ has been used. Considering the fact that the method of interpolating coverage functions can increase accuracy only by adding to interpolation functions and without the need to use finer

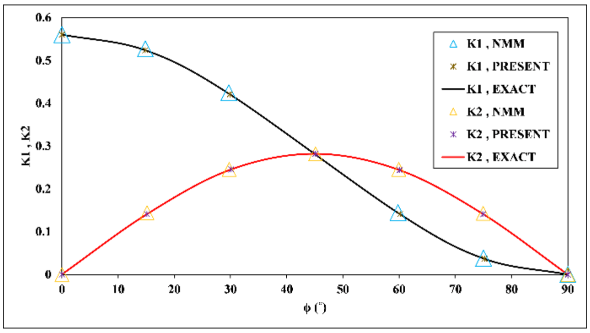

Fig. 9. KI and KII vs. β for a plate with an angled center crack

Table 2. stress intensity factors for edge crack

Methods	KI
Reference solution [11]	0.9372
XFEM Method [12]	0.954
NMM [12]	0.933
Propose method	0.938

grids, this problem is solved by 680 nodes and 16 enriched points at each tip. The results are listed in Table 1 for three cases of coverage functions of degree one, two and three where $K_I = 2.7201$ and $K_I = 2.7361$, $K_I = 2.7411$ respectively and approximation error was 1.5, 1.1 and 1.0. The comparison between the proposed method's error and that of generalized finite element's error, it is clear that the former is capable of modeling discontinuity problems with sufficient accuracy and the lower amount of calculations (fewer degrees of freedom).

5. Conclusions

To solve the discontinuity problems by finite element method, geometrical conformity is between crack and element borders is often essential. Due to limitations in grid generation, the crack is considered to be at the boundary of the grid. And thus, crack problems are not capable of being solved using a single standard mesh. Mesh generation consideration is a serious problem while modeling crack problems. The modeling process always depends on having finer mesh close the crack boundaries to cover the crack borders with elements as well as obtaining results with satisfying accuracy. But, the method of interpolating cover functions attains higher accuracy without the need to finer grid and only by increasing the degree of enrichment functions. This method is also capable of performing the analysis of discontinuity problems using a single modeling and regardless of crack and mesh considerations. This method is capable of solving the discontinuity inside the elements.

In the present paper, the method of interpolating cover functions has been used to solve the discontinuity problems such as cracks. Interpolating cover functions of degree one, two and three were used to increase the accuracy of the solution without the need to finer grids. The displacement discontinuity in the crack surface is exerted by Heaviside function. Enrichment functions such as those used in XFEM method are taken advantage of to analyze the technical situation at the tip of crack's vicinity and to compare the proposed method and with other methods. Three standard examples are investigated, and the stress coefficient has been calculated. The results show the high ability of the proposed method in solving the crack problems. It is noteworthy that the method's predominance over other methods is judged only by comparing the degrees of freedom, due to lack of comprehensive data on other investigations.

The ability to model complex geometries is among other advantages of this method, which affects in increasing accuracy and decrease in computational costs. In the proposed method, the discontinuity simulator functions are added to interpolating cover functions so that the problem containing discontinuous field is solved. And thus, a single grid can be used to solve the problem, and there will be no need to use a finer grid to increase accuracy. A rather coarser gird is sufficient to solve the problem which leads to a decrease in degrees of freedom and thus an increase in computational costs.

The results show proper accuracy in 2-dimensional problems. In all solved problems, the coefficient of stress intensity calculated by the method was close to the exact solution and other solutions given in other references. The amount of relative error was less than a percent.

REFERENCES

[1] Liu, WK., Jun, S., Zhang, YF. 1995. "Reproducing Kernel Particle Methods". *Int J Numer Methods,* Eng; 20:1081–106.

[2] Strouboulis, T., Copps, K., and Babuska, I., 2000. "The Generalized Finite Element Method: An Example of its Implementation and Illustration of its Performance". *Int J Numer Methods Eng,*17–47:1401 .

[3] Shi, GH, Goodman, RE., 1989. "Generalization of Two-Dimensional Discontinuous Deformation Analysis for Forward Modeling". *Int J Numer Anal Methods Geomech,* 80–13:359.

[4] Ghasemzadeh, H., Ramezanpour, M.A., and Bodaghpour, S., 2014. "Dynamic High Order Numerical Manifold Method Based on Weighted Residual Method". *International Journal for Numerical Methods in Engineering Int. J. Numer,* Meth. Engng, Published online in Wiley Online Library.

[5] Zheng, H., Xu, D., 2014. "New strategies for some issues of numerical manifold method in simulation of crack propagation". *Int. J. Numer. Meth. Engng,* 97, 986–1010.

[6] Nicolas, M., John, D., and Belytschko, T., 1999. "A Finite Element Method for Crack Growth without Remeshing". *International Journal for Numerical Methods in Engineering,* Int. J. Numer. Meth. Engng. 46, 131-150. 6

[7] Kim, J., Bathe, K. J., 2013. "The Finite Element Method Enriched by Interpolation Covers". *Comput Struct,* 116, pp. 35–49.

[8] Kim, J., Bathe, K. J., 2014. "Towards a Procedure to Automatically Improve Finite Element Solutions by Interpolation Covers ". *Computers & Structures,* 131, pp. 81-97.

[9] Zhang, H.H., Zhang, S.Q., 2010. "Numerical Analysis of 2-D Crack Propagation Problems Using the Numerical Manifold Method". *Finite Elements in Analysis and Design,* 59, 55–65.

[10] Belytschko, T., Black, T., 1999. "Elastic Crack Growth in Finite Elements with Minimal Remeshing". *International Journal for Numerical Methods in Engineering,* 45(5):601, 620.

[11] Arzani, H., Khoshbavar Rad, E., 2017. "Adaptive finite element analysis using h-refinement and interpolation cover functions". *sharif journal civil engineering.*

[12] Mousavi, S., E., Grinspun, E., and Sukumar, N., 2010. "Harmonic Enrichment Functions: A Unified Treatment of Multiple, Intersecting and Branched Cracks in the Extended Finite Element Method". *International Journal for Numerical Methods in Engineering,* Int. J. Numer. Meth. Engng.

HOW TO CITE THIS ARTICLE

H. Arzani, E. Khoshbavar Rad, The Analyzing of the Discontinuity Problem by Enriched Interpolation Covers, Amirkabir J. Civil Eng., 52(2) (2020) 93-98.

DOI: 10.22060/ceej.2019.13840.5487