Numerical Solution of Steady Incompressible Turbulent Navier–Stokes Equations using Multiquadric Radial Basis Function (MQ-RBF) Method

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ABSTRACT

The inconveniences of introducing and modifying the mesh grids in mesh-based numerical methods lead the researchers to meshfree methods among which the RBF methods are probably the most interesting and powerful ones. In this research the numerical solution of the steady-state incompressible continuity and Navier–Stokes equations, and the standard k-Ɛ turbulence model was investigated in a 2D domain. The computational domain consisting of a 0.5 m×0.5 m square lid-driven cavity was analyzed for five Reynolds numbers of 2.5×10^5, 5×10^5, 10×10^5, 2×10^6, and 5.5×10^6. The Multiquadric Radial Basis Function (MQ-RBF) as the most successful RBF was employed with 36 and 121 domain computational nodes to solve the PDEs. The velocity fields in two directions, the static pressure, the turbulent kinetic energy and the turbulent energy dissipation were computed. A try–and–error algorithm was used for solving a set of nonlinear equations, and the optimal values of the shape parameter *c* and the *λ* set coefficients were evaluated and discussed for each flow field. According to the results, assuming the independence of the values of the shape parameter *c* for each flow field at different Reynolds numbers, a predictable pattern can be obtained for the *λ* set for different Reynolds' numbers in the studied range. These patterns with the predictor functions of the flow fields were compared to existing benchmark results of the finite volume method (ANSYS Fluent). The Nash-Sutcliffe coefficients of 93-99% and RRSME of about %1 obtained from this comparison indicated the reasonable accuracy of the assumption concerning the independence of the shape parameter *c* of the Reynolds' numbers, the repeatable patterns of the normalized *λ* set, and polynomial predictor functions in the MQRBF method for each flow field.

KEYWORDS

Multiquadric Method, Navier-Stokes, Turbulent Flow, Lid Driven Cavity, CFD.

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1. Introduction

The fluid flow analysis using continuity equation, Navier-Stokes equations, and turbulence mathematical models has numerous applications in engineering sciences. The application of Multiquadric Radial Basis Functions (MQRBF) for solving Partial Differential Equations (PDEs) is one of the famous and efficient meshless methods. In MQRBF method, the PDEs solving procedure consist of estimating two important quantities: the shape parameter (c) and the set of unknown coefficients (λ) [1-3]. These two parameters are optimized when their resulting fields exhibit good accuracy compared to other numerical methods or experimental models. In solving the system of nonlinear PDEs including the transport equations, several shape parameters and the optimal set of coefficients must be estimated so that the solution complexity will be increased. In the present study, the set of continuity equation, Navier-Stokes equations, and mathematical turbulence model (k-Ɛ model) are analyzed assuming incompressible steady-state flow conditions consisting of five transport equations including different flow parameters and some non-linear and high-order PDE terms.

2. Methodology

The continuity and Navier-Stokes equations are applied for two-dimensional incompressible steady-state flow in isothermal conditions. Also, the k-Ɛ turbulence model with two transport equations is applied to analyze the flow turbulence parameters in high Reynolds numbers [4-5]:

$$\rho \frac{\partial \overline{u_i}}{\partial x_i} = 0 \tag{1}$$

$$\rho \frac{\partial \overline{u_i}\,\overline{u_j}}{\partial x_j} = -\frac{\partial \overline{p}}{\partial x_i} + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right) - \rho \frac{\partial \overline{u_i' u_j'}}{\partial x_j} \tag{2}$$

$$\rho \frac{\partial \overline{u_i} k}{\partial x_i} = \frac{\partial}{\partial x_j}\left((\mu + \mu_t)\left(\frac{\partial k}{\partial x_j}\right)\right) - \rho \overline{u_i' u_j'} \frac{\partial \overline{u_j}}{\partial x_i} - \rho \varepsilon \tag{3}$$

$$\rho \frac{\partial \overline{u_i} \varepsilon}{\partial x_i} = \frac{\partial}{\partial x_j}\left(\left(\mu + \frac{\mu_t}{1.3}\right)\left(\frac{\partial \varepsilon}{\partial x_j}\right)\right) + 1.44 \frac{\varepsilon}{k}\left[-\rho \overline{u_i' u_j'} \frac{\partial \overline{u_j}}{\partial x_i}\right] - 1.92 \rho \frac{\varepsilon^2}{k} \tag{4}$$

In the above equations, x represents the component of the coordinate system in principal directions, $\overline{u}$ are the mean velocity vector components, $\overline{u_i' u_j'}$ is the Reynolds stress tensor, ρ and μ are the density and dynamic viscosity, $\overline{p}$ is the static pressure, k is the turbulence kinetic energy, μ_t is the turbulent dynamic viscosity and ε is the turbulent kinetic energy dissipation [6-8]. For solving the nonlinear PDEs using MQRBF method, the following estimation function form is considered for all five domain parameters of the PDEs [9-12]:

$$f\left(x_{1_i}, x_{2_i}\right) = \sum_{j=1}^{n} \lambda_j^f \sqrt{\left(x_{1_i} - x_{1_j}\right)^2 + \left(x_{2_i} - x_{2_j}\right)^2 + c_f^2} \tag{5}$$

where x_1 and x_2 are the components of the coordinate system, c is the shape parameter, λ is the unknown coefficient, and n is the number of center points in the domain. The first and second-order derivatives of the main flow fields are derived, then the obtained MQ form of the derivatives are substituted in five transport equations (1 to 4) which result in the system of nonlinear equations. Due to nonlinear terms in the system of equations, a combination of try and error and Newton methods will be employed as the solution procedure. The results are compared with the existing benchmarks of the finite volume method using two well-known error criteria of Nash-Sutcliffe and Relative Root-Mean-Square Error for evaluating the computations accuracy.

3. Discussion and Results

The problem is solved for two cases of a lid-driven cavity benchmark with 36 and 121 center points and a sudden expansion problem with 342 center points. The results of the solved examples show that assumptions of independence of shape parameter c and predictability of λ coefficients are acceptable. Two predictor relations for λ coefficients regions based on the lid velocity U and Reynolds number are as follow in which the a_i coefficients are to be determined:

$$\underset{Min}{\overset{Max}{}}\lambda = a_3 U_{Re}^3 + a_2 U_{Re}^2 + a_1 U_{Re} + a_0 \qquad n = 3 \tag{6}$$

$$\underset{Min}{\overset{Max}{}}\lambda = a_2 U_{Re}^2 + a_1 U_{Re} + a_0 \qquad n = 2 \tag{7}$$

In most of the predicted flow fields, the results found to be in good agreement with those of the FVM (ANSYS Fluent). The Nash-Sutcliffe coefficients of 93-99% and RRSME of about %1 indicated the reasonable accuracy of the assumption concerning the independence of the shape parameter c of the Reynolds' numbers, the repeatable patterns of the normalized λ set, and polynomial predictor functions in the MQRBF method for each flow field (Table 1).

Table 1. The Nash-Sutcliffe coefficient and RRMSE error criteria for flow parameters of lid driven cavity

	Points	u_1	u_2	p	k	ε
N-S	**36**	0.97	0.96	0.97	0.99	0.98
	121	0.96	0.80	0.97	0.95	0.98
RRMSE	**36**	%1.12	%0.93	%0.37	%0.20	%0.81
	121	%0.39	%0.55	%0.08	%0.20	%0.25

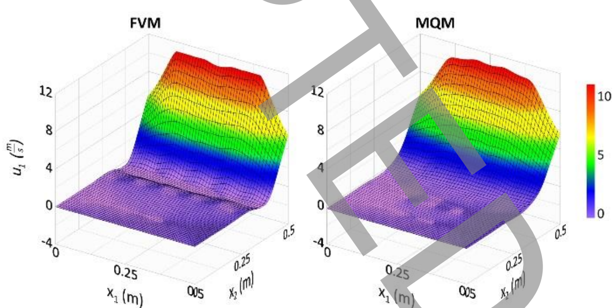

Figure 1. u_1 velocity component in lid driven cavity for MQM and FVM

4. Conclusions

In this study, the MQRBF meshfree method was examined for solving the governing equations of 2D incompressible turbulent steady flow in a lid-driven cavity benchmark problem by comparing the results to those of FVM (ANSYS Fluent). The main challenge is to find the appropriate shape parameters (c) and set of unknown coefficients (λ) for the selected number of center points. The hypothesis of shape parameters independence from Reynolds numbers and predictability of λ coefficients was validated and the high accuracy of results was indicated in addition to presenting some predictor polynomial relations for λ coefficients. The comparison between the results of the presented approach and those obtained by the FVM shows that the proposed technique may be applied solving the PDEs of 2D steady turbulent incompressible flow.

5. References

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