Amirkabir Journal of Civil Engineering

Amirkabir J. Civil Eng., 50(6) (2019) 325-328
DOI: 10.22060/ceej.2017.12265.5169

Stability and Free Vibration Analyses of Non-prismatic Columns using the Combination of Power Series Expansions and Galerkin's Method

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ABSTRACT: As a first endeavor, a mixed power series expansions and Galerkin's method in the context of linear buckling and free vibration analyses of non-uniform beams is presented. For this aim, the governing equilibrium and motion equations are first obtained from the stationary condition of the total potential energy. The power series approximation is then applied to solve the fourth order differential equilibrium equation, since in the presence of variable cross-section, geometrical properties are variable. Regarding aforementioned method, the expression of deflected shape of the buckled member is identified. Afterwards, the critical buckling loads can be acquired by imposing the boundary conditions and solving the eigenvalue problem. Note that the buckling mode shapes of an elastic member are similar to the vibrational ones. Therefore, the obtained deformation shape of the considered non-prismatic columns under linear stability analysis can be used as vibrational shape of member. The natural frequencies of beams with varying cross-section can be estimated by adopting Galerkin's method based on the energy principle. In order to illustrate the correctness and performance of this method, one comprehensive example of non-uniform beams with various end conditions is presented.

Review History:

Received: 1 January
Revised: 1 August 2017
Accepted: 9 March 2016
Available Online: 23 October 2017

Keywords:

Critical Buckling Load
Natural Frequency
Non-prismatic Column
Power Series Method
Galerkin's Method

1- Introduction

Due to improvements in fabrication process, flexural members with variable cross-section, known as non-prismatic beam, are extensively spread in different engineering structures such as high-rise buildings, aeronautical structures, cranes and other application fields. Researchers use various methods especially numerical ones to solve the equilibrium and motion equations of non-prismatic elements because of its relevance civil and mechanical engineering. The first investigation in this field was presented by Euler. He studied the critical buckling load of columns under their own weight. Timoshenko [1] derived the governing equilibrium equations and their closed-form solutions for various types of flexural members under different circumstances. Arbabi and Li [2] presented a semi analytical approach for measuring buckling load of columns with step-varying profiles. Rahai and Kazemi [3] formulated a new approach for the problem of buckling of tapered column members. The exact buckling load is calculated by combination of modified vibrational mode shape (MVM) and energy method. Coşkun and Atay [4] used variational iteration method to determine the critical buckling load of elastic columns with variable cross sections. Okay et al. [5] found buckling loads and mode shapes of a heavy column by applying the variational iteration method.

The abovementioned researchers investigated the problem of buckling and natural frequency only for special types of columns. The main purpose of this paper is calculating the critical buckling loads and natural frequencies for any types of members with linear or polynomial variation of cross-sectional profile based on the power series expansions combined with Galerkin's method. The followings are the abstract gist of this paper:

1. In order to ease the solution of the governing equilibrium differential equation of non-prismatic member with variable coefficients, the power series expansions are applied. Regarding this, it can be noticed that the function describing the moment inertia of the beam is expanded into power series form. The critical buckling loads of the member were derived by imposing the boundary conditions and solving the eigenvalue problem. The explicit expression of buckled shape function is thus acquired based on this rigorous numerical method.
2. Based on the similarities between vibration and buckling deformation shapes of elastic members, besides; adopting Galerkin's method based on the principle of stationary total potential energy along the beam axis, the natural frequency of considered non-uniform beam is also evaluated.

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Finally, for measuring the accuracy and validity of the proposed procedure, one comprehensive numerical example is represented. The outcomes are also compared with other accessible results. This method has many positive points consisting of efficiency, accuracy and simplicity contrasted with more complex numerical methods.

2- Formulation

A non-prismatic beam of length L as depicted in Figure 1 is taken into account. Following Euler-Bernoulli beam theory, the equilibrium equation for non-prismatic member subjected to a constant axial load can be expressed as:

$$\frac{d^2}{dx^2}\left[EI_y(x)\frac{d^2w}{dx^2}\right]-P\frac{d^2w}{dx^2}=0 \tag{1}$$

In the last formulation, I_y is the minor-axis moment of inertia which can be arbitrary over the beam's length (x-axis). The displacement components (w) represents vertical deformation (in z direction). E denotes Young's modulus of elasticity for a homogeneous and isotropic material.

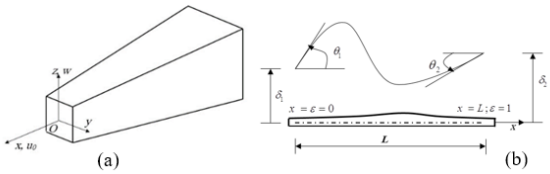

Figure 1. (a) A non-prismatic beam (Coordinate system and notation of displacement parameters), (b) Boundary conditions of a beam element in global and local coordinate system

In order to make the solution of the stability equation easy, a non-dimensional variable (ε= x/L) is introduced. Regarding the power series method, all the variable terms in Equation 1, namely, moment of inertia and the displacement parameter, should be presented in power series form, as follows:

$$I_y(\varepsilon)=\sum_{i=0}^{\infty}I_i^*\varepsilon^i \qquad w(\varepsilon)=\sum_{i=0}^{\infty}a_i\varepsilon^i \tag{2}$$

In which:

$$I_i^*=I_iL^i \tag{3}$$

By substituting Equation 2 into Equation 1 and after some required algebra, the following recurrence formula about a_{j+4} is obtained:

$$a_{j+4}=\frac{-1}{EI_0^*(j+4)(j+3)(j+2)(j+1)}\Big\{PL^2(j+1)(j+2)a_{j+2} +E\sum_{i=1}^{j+2}I_i^*(j-i+4)(j-i+3)(j+2)(j+1)a_{j-i+4}\Big\} \tag{4}$$
$$for \quad j=0,1,2,.....$$

According to the acquired recurrence formula and from mathematical point of view, it is culminated that all the a_i coefficients except for the first four (a_0, a_1, a_2, a_3) can be obtained. The general solution of Equation 1 can be thus expressed in the following form:

$$w(\varepsilon)=a_0w_0(\varepsilon)+a_1w_1(\varepsilon)+a_2w_2(\varepsilon)+a_3w_3(\varepsilon) \tag{5}$$

All terms of the fundamental solutions of the equilibrium equation ($w_i(\varepsilon)$, i=0, 1, 2, 3) are derived with the aid of the symbolic software MATLAB [6]. According to Figure 1b, the boundary conditions at the left end of the beam (ε=0) can be written as:

$$1)\ \text{At}\quad x=0\rightarrow\varepsilon=0\Rightarrow w(0)=\delta_1 \tag{6}$$

and

$$2)\ \text{At}\quad x=0\rightarrow\varepsilon=0\Rightarrow \left.\frac{dw(x)}{dx}\right|_{x=0}=\theta_1=\frac{1}{L}\left.\frac{dw(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} \tag{7}$$

Referring to Figure 1b and using local coordinates, the following boundary conditions at the right end (ε=1) of the given elastic member exist:

$$3)\ \text{At}\quad x=L\rightarrow\varepsilon=1\Rightarrow w(L)=w(1)=\delta_2 \tag{8}$$

and

$$4)\ \text{At}\quad x=L\rightarrow\varepsilon=1\Rightarrow \left.\frac{dw(x)}{dx}\right|_{x=L}=\theta_1=\frac{1}{L}\left.\frac{dw(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=1} \tag{9}$$

According to the abovementioned end conditions, a general system of four linear equations with four unknowns (a_0, a_1, a_2, a_3) is derived. The function defining the deformation shape of the non-prismatic beam under linear stability analysis is finally obtained at any places of member as:

$$\begin{aligned} w(\varepsilon)&=\delta_1w_0(\varepsilon)+L\theta_1w_1(\varepsilon)\\ &-\frac{1}{C}\left\{w_3'(1)\left(\delta_2-\delta_1-L\theta_1\right)-w_3(1)\left(L\theta_2-L\theta_1\right)\right\}w_2(\varepsilon)\\ &+\frac{1}{C}\left\{w_2'(1)\left(\delta_2-\delta_1-L\theta_1\right)-w_2(1)\left(L\theta_2-L\theta_1\right)\right\}w_3(\varepsilon)\end{aligned} \tag{10}$$

In which

$$C=w_3(1)w_2'(1)-w_2(1)w_3'(1) \tag{11}$$

By pondering on the last formulation, the roughly function of buckling mode shape can be determined in terms of boundary conditions of a beam element. In this regard, let us consider fixed-hinged member in which the transverse displacement at both ends and rotation at the fixed end (x=ε=0) are prevented. The explicit expression of buckled shape for fixed-hinged members is thus determined as:

$$w(\varepsilon)=\frac{L}{C}(w_3(1)w_2(\varepsilon)-w_2(1)w_3(\varepsilon))\theta_2 \tag{12}$$

It is culminated that the approximate expression for vertical displacement in the non-dimensional coordinate can be finally derived in terms of rotation of right end of member (θ_2), which can be written as:

$$w(\varepsilon) = \theta_2 G(\varepsilon) \tag{13}$$

To carry out free transverse frequencies using Galerkin's method which is directly applied to the differential equation, an appropriate deformation shape of the element after flexural buckling satisfying the geometric and natural boundary conditions of the system is required. Based on the similarities between vibration and buckled deformation shapes of elastic members, the obtained buckling mode (Equation 12) can be thus adopted as deformed shape of column for the free vibration analysis. The bending free vibration can be thus calculated by the following expression:

$$\int_0^1 \left(\frac{d^2}{d\varepsilon^2}(EI_y(\varepsilon)\frac{d^2G}{d\varepsilon^2}) + \omega^2 \rho A(\varepsilon) G(\varepsilon) \right) G(\varepsilon) d\varepsilon = 0 \tag{14}$$

3- Numerical Example:

In this example, the free vibration and stability analyses of three non-prismatic columns, as shown in Figure 2, with different boundary conditions are investigated. Each column has a rectangular cross-section. In all considered cases, the geometrical properties of the fixed end section of the member are constant. The depth of the column is made to reduce to half at the top of the member with a parabolic variation, while its width remains constant. The aim of this example is also to define the required number of terms in power series expansions to obtain an acceptable accuracy on critical buckling loads and natural frequencies of non-prismatic members. The distribution of moment of inertia I(ε) and cross-sectional area A(ε) of the considered section in non-dimensional coordinate are described as follows:

$$\begin{aligned} I(\varepsilon) &= I_A\left(1-0.5\varepsilon^2\right) \\ A(\varepsilon) &= A_A\left(1-0.5\varepsilon^2\right) \end{aligned} \tag{15}$$

Effect of the number of power series terms (N) considered in the proposed numerical technique on convergence is also displayed in Table 1. The dimensionless buckling parameter is acquired as:

$$P_{cr} = \lambda_{cr} \frac{\pi^2 EI_B}{L^2} \tag{16}$$

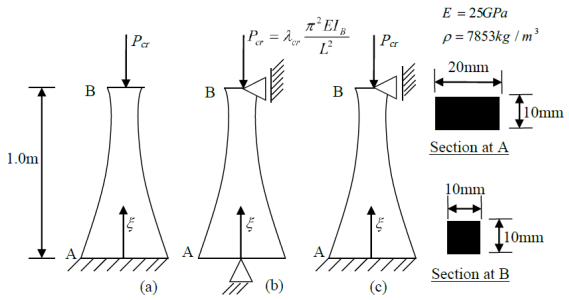

Figure 2. Non-prismatic columns with different boundary conditions. (a) Clamped-Free (C-F); (b) Pinned-Pinned (P-P); (c) Clamped-Pinned (C-P)

Table 1. Buckling load parameter (λ_{cr}) of power series method and finite element results for non-prismatic beams with different boundary conditions

Boundary Conditions	Number of terms of power series (N)					Ansys [7]
	10	20	30	40	50	
C-F	0.404	0.459	0.463	0.463	0.463	0.463
P-P	1.713	1.688	1.688	1.688	1.688	1.686
C-P	3.279	3.349	3.349	3.349	3.349	3.348

According to Table 1, it can be concluded that for high accurate solution involved in the stability analysis, it is not required to take more than 30 terms of power series by using proposed method. Table 2 gives the lowest values of natural frequency of non-prismatic beams with different boundary conditions, as illustrated in Figure 2.

Table 2. Natural frequencies comparison of power series method and finite element results for non-prismatic beams with different boundary conditions

Boundary Conditions	Present Method	Ansys [7]
C-F	63.47	63.48
P-P	146.77	146.81
C-P	234.83	234.92

One can observe from Table 2 that 30 terms in the power series expansion is a good compromise for equivalent accuracy of the both buckling loads and natural frequencies. The competency and efficiency of the proposed numerical method is also remarked

4- Conclusions

In this paper, the power series method is simultaneously adopted with Galerkin's approach and applied to non-prismatic beams having generalized end conditions. The power series approximation is used to solve the equilibrium equation of beam with non-uniform cross-section. Regarding

this numerical method, displacement component and cross-section properties are expanded in terms of power series of a known degree. By solving the eigenvalue problem, one can acquire the critical buckling loads. According to aforementioned method, the expression of buckling mode is also determined. Based on the similarities existed between the vibrational and buckling deformation shapes, the natural frequencies of considered member can be evaluated by applying the buckling mode in the motion equation instead of vibrational one.

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Please cite this article using:

M. Soltani, B. Asgarian, Stability and free vibration analyses of non-prismatic columns using the combination of power series expansions and Galerkin's method, *Amirkabir J. Civil Eng.*, 50(6) (2019) 1017-1032.

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