A comparison between the performance of RBF-MQ method and cell-centered finite volume method in solving 2d wave equation

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ABSTRACT

In this study, the performance of the multi-quadratic radial basis function (RBF-MQ) method is compared with the cell-centered finite volume method (FVM-CC) for solving the two-dimensional second-order wave equation. The wave equation describes wave propagation, such as sound waves, and relates the second derivatives in time and space to the wave speed. The finite volume method and radial basis function method are among the techniques used for modeling partial differential equations, such as the wave equation, in complex geometries. To compare the efficiency of these two methods, the finite volume method is first formulated based on triangular cell-centered volumes. The radial basis function superposition method is also developed using a multiquadric function for scattered points. Finally, time discretization is carried out using the explicit Euler method. The comparison of the performance and capabilities of both methods is made by performing several numerical tests on different geometries, initial and boundary conditions, scattered points, shape variables, and processing time for each method. For a more quantitative comparison, an analytical solution of the wave equation is obtained using the method of separation of variables for a simple rectangular geometry with specified initial and boundary conditions. The results of this study indicate lower computational cost and better efficiency of the multiquadric radial basis function method compared to the cell-centered finite volume method. However, the RBF-MQ method strongly depends on the shape parameter and the selected scattered point distribution. Additionally, the FVM-CC method provides higher relative accuracy compared to RBF-MQ, regardless of the shape parameter and selected scattered point distribution.

KEYWORDS

Two-dimensional wave equation, finite volume method, multiquadric radial basis function method, cell-centered method, meshless method.

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1. Introduction

Partial Differential Equations (PDEs) are widely used in physics, engineering, finance, biology, and other fields to model various phenomena. PDEs can be classified into elliptic, parabolic, and hyperbolic, depending on their properties such as diffusion, or wave-like behavior. Hyperbolic equations are suitable for simulating dynamic phenomena with wave-like behavior like flow propagation. Examples include Euler equations, shallow water equations, and classical wave equations. Due to the complexity of the 2d hyperbolic equations, numerical methods like finite difference and finite volume are often used for their solution. Among these, the finite volume method such as the cell-centered approach, is common for solving hyperbolic equations, offering flexibility in dealing with complex geometries [1-2]. Meshless methods like the radial basis function (RBF) method have recently gained attention for their simplicity and low computational cost, representing the domain using points and solutions using RBFs. However, meshless methods face solvability issues and heavily rely on the shape parameter and distribution of collocation points [3-4]. This article aims to quantitatively compare the costs, accuracy, and efficiency of RBF-MQ and FVM-CC methods for solving 2d wave equations. The study includes analyzing the behavior and performance of meshless and mesh-based methods, estimating accuracy using L_2 and L_∞ error norms, sensitivity to the shape parameter and the distribution of data in complex geometries, and calculating CPU runtime for each method.

2. Methodology

The finite volume method can be derived through the integral form of the conservation law. In this study, for numerical discretization of the domain, the cell-centered technique with structured and unstructured triangular grids is used (Figure 1). The cell-centered approach is known for its better performance when compared to the node-centered one. To determine the numerical fluxes at the cell boundaries, the second-order 2d wave equation is rewritten as a system of two first-order partial differential equations in 2d form. Then a first-order FVM-CC scheme based on a triangular cell-centered grid is used for solving the PDEs. The radial basis function collocation method established here utilizes multiquadric RBFs for scattered nodes (Figure 1). To develop the RBF method for the spatial term, a semi-discrete approach is utilized. The semi-discrete equation is interpolated using a basis function, and then the estimated values are updated in subsequent time steps by applying an explicit time marching scheme. Interpolation using RBFs provides higher accuracy and flexibility compared to other basis functions such as least squares methods. Radial basis functions commonly used include Gaussian, thin plate spline, multiquadric, and inverse multiquadric functions. Among these functions, the multiquadric function is employed in this study due to the higher accuracy. For both the cell-centered finite volume method and the multiquadric radial basis function developed here, the temporal discretization is then performed using the explicit Euler scheme. Moreover, for quantitative comparison, an analytical solution of the wave equation for a simple rectangular geometry with specified initial and boundary conditions is obtained using the method of separation of variables.

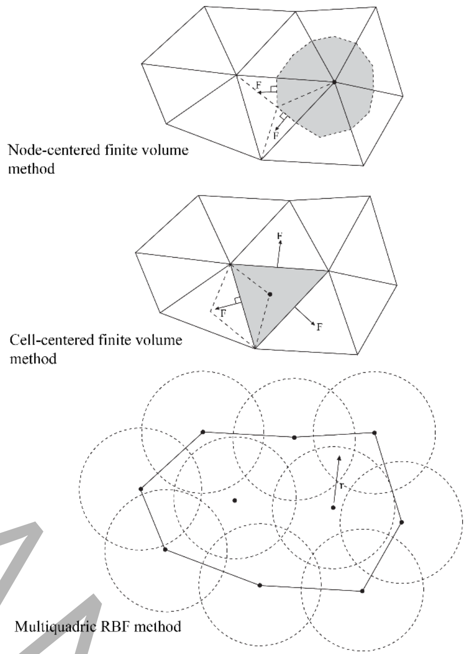

Figure 1. Discretization of node-centered finite volume method, cell-centered finite volume method, and multiquadric radial basis function method.

3. Results and Discussion

To compare the efficiency and the accuracy of the FVM-CC and RBF-MQ, several numerical tests for various geometries and different initial and boundary conditions, considering the distribution of scattered data, shape parameters, and processing runtimes are conducted. The first experiment concerns the quantitative comparison of both methods with the analytical solution of the wave equation obtained using the method of separation of variables. For this experiment, a square-shaped domain is used. Two Gaussian and sinusoidal-like initial conditions are applied and Dirichlet boundary type with zero values is imposed. Table 1 presents numerical errors and computational times for FVM-CC and RBF-MQ. The

FVM-CC exhibits higher error compared to the finite difference method (on Cartesian grid) and RBF-MQ (with optimal shape parameter). The RBF-MQ's sensitivity to the shape parameter is shown; for certain shape parameter values, its error exceeds FVM-CC. The stability and accuracy of RBF-MQ depend on the shape parameter. The FVM-CC's accuracy is affected by cell-centered value approximation, making it less accurate than FDM. Its stability, however, relies solely on the Courant number. The FVM-CC displays higher computational costs than FDM and RBF-MQ.

Table 1. Numerical errors and performances of CC finite volume and radial basis function methods.

Test cases	method	Numerical errors and performances	
		MAPE %	CPU runtime (s)
1-a	FDM-LF	6.94	0.025
	FVM-CC	8.96	1.144
	RBF-MQ	9.70	0.109
1-b	FDM-LF	5.07	0.038
	FVM-CC	6.98	1.541
	RBF-MQ	7.74	0.156
2	FVM-CC	5.41	1.549
	RBF-MQ	6.42	0.172
3	FVM-CC	6.07	1.527
	RBF-MQ	7.68	0.167
4	FVM-CC	5.52	1.119
	RBF-MQ	7.02	0.122

For further experimentation of these methods in complex domains, the flow field in a two squared-shape with cut-outs and annulus membranes, is modeled. Additionally, for better comparison, a reference solution using a finer mesh is generated. The accuracy and computational costs for each method are reported in Table 1. The runtime for the FVM-CC is higher compared to RBF-MQ, indicating lower computational costs for the latter. The accuracy of RBF-MQ depends on the chosen shape parameter.

4. Conclusion

The results of this study indicate that the RBF-MQ leads to lower computational costs and consequently exhibits better efficiency compared to the FVM-CC. However, the RBF-MQ is highly dependent on the shape parameter and the distribution of collocation points. Overall, the FVM-CC shows relatively better accuracy compared to RBF-MQ regardless of the shape parameter and the selected collocation points.

5. References

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