Development of Gauss-Legendaire-Hermite-3Point (GLH-3P) Formulation for Linear and Nonlinear Analysis of Earthquake-Affected Structures

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ABSTRACT

The dynamic response of structures under seismic loading is a critical issue in civil engineering that requires precise and efficient analytical methods. This research presents an efficient formulation for the nonlinear dynamic analysis of structures termed the Gauss-Legendre-Hermite Three-Point (GLH-3P) method. This method is based on the three-point implicit Gauss integration rule and employs third-order Hermite interpolation for sub-step approximations. The proposed formulation is capable of analyzing systems with geometric and material nonlinearities and covers various loading patterns. Results obtained from the new method were compared with established techniques such as the semi-analytical Duhamel integral and the pseudo-analytical Newmark-beta and Wilson-theta methods. The results indicate that the proposed method reduces the root mean square (RMS) error by up to 18% in linear systems and up to 93% in nonlinear systems compared to the Newmark method. The formulation demonstrates significant superiority in terms of accuracy, stability, convergence, and computational cost.

KEYWORDS

Time-history analysis, Nonlinear analysis, Newmark method, Gauss-Legendre quadrature, Hermite interpolation.

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1. Introduction

The oscillatory motion of an ideal mass-spring-damper system is modeled by a second-order ODE[2] known as the DEOM[3]. To predict the behavior of this system over time, the DEOM must be solved within a specific time interval. While analytical solutions are generally preferred, they are often difficult to obtain in practical engineering due to irregular time-varying excitation functions (e.g., earthquakes) and nonlinear system behavior [1].

Step-by-step time integration methods provide a comprehensive framework for analyzing both linear and nonlinear systems. Traditional methods such as Central Difference, Newmark-beta, Houbolt, and Wilson-theta are widely used in structural dynamics [2, 5]. However, developing more efficient formulations that provide higher accuracy with larger time steps remains a persistent challenge for researchers [3].

The main objective of this study is to generalize the Gauss-Legendre integration method and combine it with Hermite interpolation formulas to create a novel approach for solving the equation of motion for SDOF[4] systems. The innovation lies in the development of specific Hermite interpolation coefficients and an optimized iterative algorithm that increases the order of accuracy to $O(h^6)$, whereas standard methods like Newmark typically provide $O(h^2)$ accuracy [4].

2. Methodology

The formulation of the proposed GLH-3P[5] method requires four mathematical tools: time-domain discretization, Gauss-Legendre integration, Hermite interpolation formulas, and Taylor series predictors.

2.1. Time-Domain Discretization

The continuous time domain is discretized into N steps of length $h = \Delta t$. The discrete form of the equation of motion at time t_{i+1} is expressed as: The basic elements of the extended abstract are listed below in the order in which they should appear:

$$m\ddot{u}_{i+1} + f_D(\dot{u}_{i+1}) + f_S(u_{i+1}) = P_{i+1} \qquad (1)$$

Where m is mass, f_D is damping force, and f_S is restoring force.

2.2. Gauss-Legendre-Hermite Integration

The core of the algorithm involves calculating the response at three internal Gaussian points within each time step. These points are defined as τ_1, τ_2 and τ_3 based on the three-point Gauss-Legendre rule. The geometric representation of this method is illustrated in Figure 1, which shows the distribution of Gaussian points within the normalized time interval.

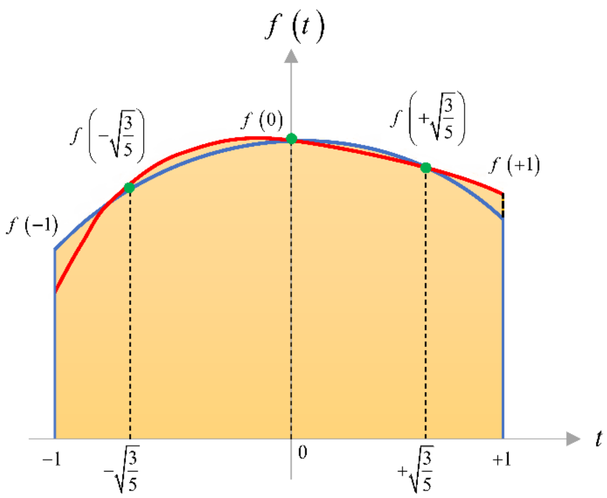

Figure 1. Schematic diagram of the three-point Gauss-Legendre method for sub-step integration

2.3. Hermite Interpolation Coefficients

Unique coefficients for velocity and displacement interpolation at internal sub-steps were developed for the first time in this study. These coefficients, based on function approximation theory, allow for high-precision estimation of the system's state using end-point information. The coefficients used in the proposed algorithm are summarized in Table 1.

Table 1. Hermite interpolation coefficients for calculating velocity and displacement at sub-steps

Velocity Interpolation coefficients		
$A = \frac{1}{2} + \frac{3\sqrt{15}}{25} \cong \frac{219}{227}$	$B = \frac{1}{2} - \frac{3\sqrt{15}}{25} \cong \frac{533}{15124}$	$E = \frac{1}{2}$
Displacement Interpolation coefficients		
$G = \frac{1}{2} + \frac{63\sqrt{15}}{500} \cong \frac{1070}{1083}$	$H = \frac{1}{2} - \frac{63\sqrt{15}}{500} \cong \frac{82}{6831}$	$M = \frac{1}{2}$
$I = \frac{11}{200} + \frac{13\sqrt{15}}{1000} \cong \frac{518}{4917}$	$J = \frac{11}{200} - \frac{13\sqrt{15}}{1000} \cong \frac{409}{87934}$	$N = \frac{5}{32}$

[2] Ordinary Differential Equation
[3] Dynamic Equation of Motion
[4] Single Degree of Freedom
[5] Gauss-Legendre-Hermite 3Point

$$K = \frac{1}{400} + \frac{\sqrt{15}}{2000} \cong \frac{62}{13975} \quad L = \frac{1}{400} - \frac{\sqrt{15}}{2000} \cong \frac{67}{118898} \quad O = \frac{1}{64}$$

Two numerical models were implemented in MATLAB to evaluate the GLH-3P method using the El Centro earthquake record (Figure 2).

3. Results and Discussion

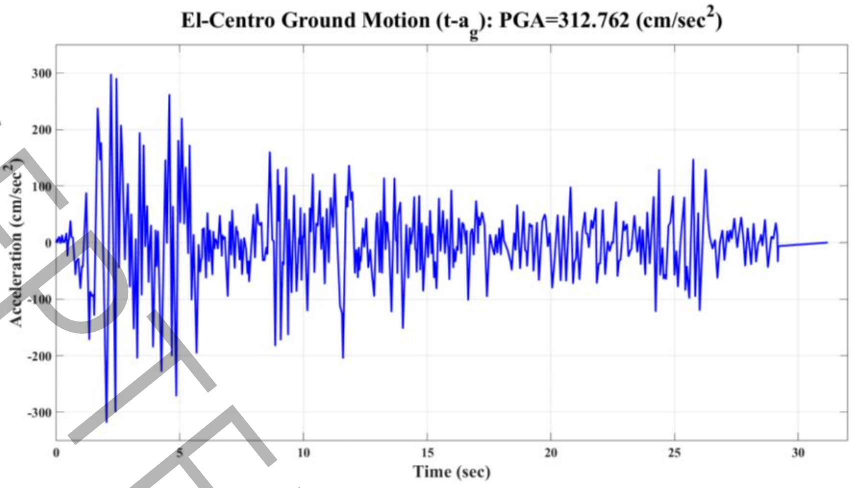

Figure 2. El-Centro earthquake record used in the loading of structural systems

3.1. Model I: Linear System

A linear undamped system with a natural period of 0.113 s was analyzed. The GLH-3P method provided results nearly identical to the exact Duhamel integral solution, whereas the Newmark and Wilson methods showed significant phase shift errors. As shown in Table 2, the RMS error for GLH-3P was significantly lower than that of conventional methods.

Table 2. Comparison of maximum response and RMS error for Model I (Linear System)

Evaluation Component	Duhamel (Exact)	GLH-3P (Proposed)	Newmark-β	Wilson-θ
Max Displacement (cm)	0.2618	0.2697	0.4612	0.4613
Max Velocity (cm/s)	5.7611	5.6972	7.1815	7.1812
RMS Error	0.0879	0.1039	0.328	0.3278

3.1. Model II: Nonlinear System

A nonlinear damped system with a natural period of 0.08 s and elastoplastic behavior with kinematic hardening was investigated. The proposed method demonstrated superior stability and accuracy in capturing the hysteresis loops compared to the Newmark method. Notably, the GLH-3P method does not require matrix inversion in the iterative cycle, reducing computational complexity for certain applications.

4. Conclusion

The GLH-3P method provides a robust and efficient framework for the time-history analysis of SDOF structures. Key advantages include:

1) Reduction of RMS error by up to 93% in nonlinear systems compared to the Newmark method.
2) A unified formulation that handles both linear and nonlinear analysis without changing the underlying algorithm.
3) High accuracy in identifying jump points and high-frequency oscillations in nonlinear response curves.
4) While the method is more computationally intensive for high-frequency systems due to its iterative nature, its precision makes it a reliable tool for seismic analysis.

5. References

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