Amirkabir Journal of Civil Engineering

Amirkabir J. Civil Eng., 54(11) (2023) 841-844
DOI: 10.22060/ceej.2022.20119.7345

Solution of 3D elasticity problems using meshless local equilibrated basis functions

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ABSTRACT: A mesh-free method is presented for 3D elasto-static problems in homogenous media using Equilibrated Basic functions. The method treats satisfaction of the Partial Differential Equation independent of the boundary conditions, using a weak weighted residual integration over a cubic fictitious domain embedding the main domain. All 3D integrals break into the combination of 1D library integrals, resulting in the omission of the numerical integration. Chebyshev polynomials of the first kind are used to approximate the solution function, and exponential functions combined with polynomials are used as weight functions. The weights vanish over the boundaries of the cubic fictitious domain, removing the boundary integrals. The meshless method considers some nodes for the definition of the Degrees of Freedom throughout the domain. Each node corresponds to a local sub-domain called cloud, including 98 other nodes than the main central one. The overlap between adjacent clouds ensures the continuity of both the displacement as well as stress components, an advantage with respect to the formulations. The approximation order within each cloud is 4. Boundary conditions are applied over a set of boundary points independent of the domain nodes, granting the method the ability of application for arbitrarily shaped domains without the drawback of irregularity in the nodal grid. The definition of curved boundary surfaces is easily done by inserting the coordinates of some boundary points located on them. Three numerical examples with various geometries and boundaries are presented to challenge the method. The results are compared with either the available exact solutions or the FEM.

Review History:

Received: Jun. 03, 2021
Revised: Jul. 12, 2022
Accepted: Aug. 14, 2022
Available Online: Aug. 27, 2022

Keywords:

Elasticity
Three dimensional
Equilibrated basis functions
Meshless method
Weighted residual approach

1- Introduction

To solve the Partial Differential Equations (PDEs) governing engineering problems, including the elasticity equations, there are various numerical modeling, among which the finite element method (FEM) is the most popular and widely used. In general, it is desired that the mesh is as ideal as possible and well-structured, since distorted geometry may have a negative impact on the solution accuracy. In order to overcome such drawbacks of mesh-based methods, meshless methods have been developed for solving PDEs in engineering and other sciences. The principal feature of meshless methods is the use of appropriate approximation schemes that can approximate the data specified on the randomly located nodes without the use of pre-defined mesh. Need for a truly meshless method leads to the development of Element Free Galerkin (EFG) method [1] and Mesh-less Local Petrov-Galerkin (MLPG) method [2]. The use of Exponential Basis Functions (EBFS) to develop a local meshfree method which uses explicit relations to satisfy only PDEs with constant coefficients was considered in [3, 4]. In 2015, the Mesh-less Equilibrated Basis Functions (MLEqBFs) were presented in [5], which were extended to solve two-dimensional problems in heterogeneous media, including FGM problems. The most important innovations of the present paper compared to the previous related studies can be stated as: the first formulation of Equilibrated Basis functions (EqBFs) for three-dimensional elasto-static problems, development of the method in local meshless form, while all previous works were in boundary form, and separation of the main domain points and boundary points, which leads to an easier definition of arbitrary geometries as well as the absence of irregularity in the nodal grid, and thus its undesirable outcomes.

2- Methodology

The general form of the equilibrium PDE is considered according to (1),

$$\mathbf{S}^T\mathbf{D}\mathbf{S}\mathbf{u}+\mathbf{b}=\mathbf{0} \quad in\ \Omega, \quad \mathbf{B}(\mathbf{u})=\mathbf{f}_b \quad on\ \Gamma \tag{1}$$

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The homogeneous part of the solution to the above PDE is set as,

$$\mathbf{u}_h \simeq \hat{\mathbf{u}}_h = \sum_{j=1}^{N} \mathbf{f}_j \mathbf{c}_j = \mathbf{f}^T \mathbf{c} \tag{2}$$

Equation (2) is not able to satisfy the homogeneous part of the PDE, therefore it should be estimated in the form of the weighted residual integral over a fictitious cubic domain,

$$\int_{\Omega_*} \mathbf{w}_i \mathbf{S}^T \mathbf{DS}(\mathbf{f}^T \mathbf{c}) d\Omega = \mathbf{0}, \qquad i = 1,\dots,M \tag{3}$$

This process will eventually lead to the formation of a matrix equation as in (4) using all weight functions,

$$\mathbf{Ac} = \mathbf{0} \qquad \mathbf{A} = \int_{\Omega_*} \mathbf{w}_i \mathbf{S}^T \mathbf{DSf}^T d\Omega \tag{4}$$

If **c** is a member of the null-space of **A**, Equilibrated Basis Functions could be extracted. So, Equation (2) converts into (5),

$$\hat{\mathbf{u}}_h = \sum_{r=1}^{\bar{M}} \mathbf{f}^T \boldsymbol{\varphi}_r d_r = \mathbf{f}^T \boldsymbol{\varphi}\mathbf{d} \tag{5}$$

Unknown coefficients **d** are found by applying the boundary conditions. Solving the problem in two parts, homogeneous solution (*h*) and private solution (*p*) is expressed in the form of (6),

$$\mathbf{u}_h^i \simeq \hat{\mathbf{u}}_h^i = \mathbf{F}_i \mathbf{d}_i = \mathbf{f}_i^T \boldsymbol{\varphi}_i \mathbf{d}_i . \quad \mathbf{u}_p^i \simeq \hat{\mathbf{u}}_p^i = \mathbf{f}_p^i \mathbf{c}_p^i . \tag{6}$$

In the above relation, the index i indicates the allocation of the homogeneous solution to a cloud-centered at the *i-th* node. Establishing a relationship between the unknown coefficients of the solution series with the degrees of freedom of the nodes within the cloud, leads to the following,

$$\mathbf{U}_E^{h,i} = \mathbf{f}_i^E \boldsymbol{\varphi}_i \mathbf{d}_i = \boldsymbol{\psi}_E^{h,i} \mathbf{d}_i , \quad \mathbf{d}_i = \left(\boldsymbol{\psi}_E^{h,i}\right)^+ \mathbf{U}_E^{h,i} \tag{7}$$

In the above, the + sign indicates the Moore-Penrose pseudo-inverse. A relationship between the central node of the cloud and the other nodes leads to,

$$\mathbf{u}_h^i\Big|_{\mathbf{x}_c} = \mathbf{U}_C^{h,i} = \mathbf{f}_i^T\Big|_{\mathbf{x}_c} \boldsymbol{\varphi}_i \left(\boldsymbol{\psi}_E^{h,i}\right)^+ \mathbf{U}_E^{h,j} \tag{8}$$

For the total solution, the contribution of the homogeneous part should be calculated by considering the contribution of the particular solution in each of the cloud nodes, which leads to,

$$\mathbf{U}_C^i - \mathbf{f}_i^T\Big|_{x_c} \boldsymbol{\varphi}_i \left(\boldsymbol{\psi}_E^{h,i}\right)^+ \mathbf{U}_E^i = \mathbf{U}_C^{p,i} - \mathbf{f}_i^T\Big|_{x_c} \boldsymbol{\varphi}_i \left(\boldsymbol{\psi}_E^{h,i}\right)^+ \mathbf{U}_E^{p,i} \tag{9}$$

For imposing the Dirichlet and Neumann boundary conditions, Equations (10) and (11) will be built at the corresponding boundary point,

$$\mathbf{T}_n \mathbf{f}_i^T\Big|_{\mathbf{x}_{Bj}-\mathbf{x}_{Ci}} \boldsymbol{\varphi}_i \left(\boldsymbol{\psi}_E^{h,i}\right)^+ \left(\mathbf{U}_E^i - \mathbf{U}_E^{p,i}\right) = \mathbf{U}_{Bj} \tag{10}$$

$$\mathbf{nDS}\,\mathbf{f}_i^T\Big|_{\mathbf{x}_{Bj}-\mathbf{x}_{Ci}} \boldsymbol{\varphi}_i \left(\boldsymbol{\psi}_E^{h,i}\right)^+ \left(\mathbf{U}_E^i - \mathbf{U}_E^{p,i}\right) = \mathbf{t}_{Bj} \tag{11}$$

By adding the boundary equations to the continuity equations, the overall matrix set is completed as follows.

$$\mathbf{KU} = \mathbf{U}_B \tag{12}$$

The rows of matrix **K** are made from Equations (9-11).

3- Results and Discussion

To show that the proposed method does not need to produce a structured nodal grid according to the geometry of the domain, a 3D elasticity problem with a hollow spherical domain subjected to inner pressure is investigated. The exact solution for stress and displacement is,

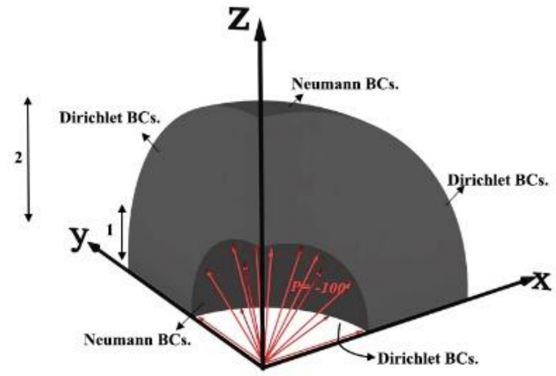

Fig. 1. Spherical problem and its boundary conditions

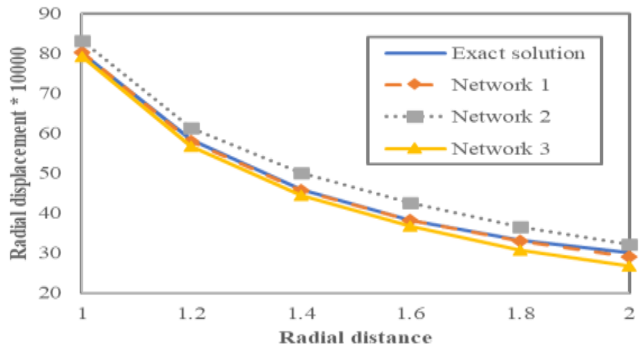

Fig. 2. Radial displacement comparison with the exact solution

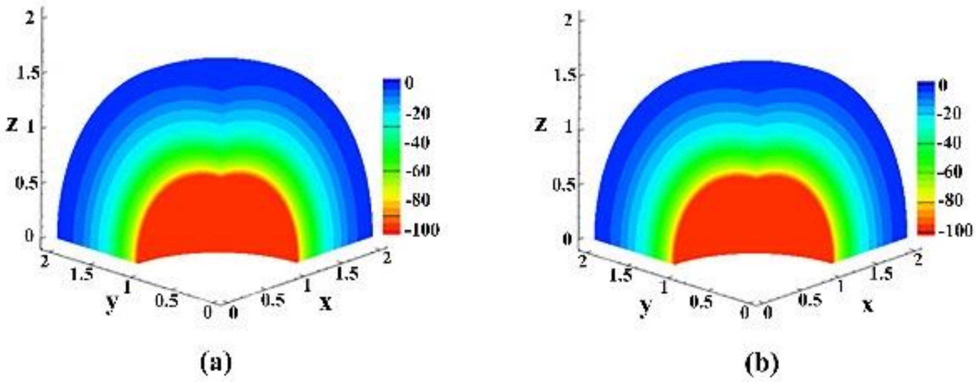

Fig. 3. Comparison of σ_r using (a) exact solution and (b) present method

$$u_r = \frac{Pa^{۳} r}{E(b^{۳} - a^{۳})}\left[(۱-۲\nu)+(۱+\nu)\frac{b^{۳}}{۲r^{۳}}\right], \qquad (13)$$
$$\sigma_r = \frac{Pa^{۳}(b^{۳}-r^{۳})}{r^{۳}(a^{۳}-b^{۳})}.$$

Due to symmetry, only 1/8 of the whole domain is considered. Boundary conditions are shown in Figure 1. The material properties are $E = 10^4 N / m^2$ and $\nu = 0.3$. The approximation order within the clouds is 4, and the number of points in each cloud is 98.In order to measure the effects of the number and the arrangement of the nodes on the accuracy of the answer, three grid cases are selected to solve this problem, Grid-1: 912 nodes uniformly distributed, Grid-2: 641 nodes non-uniformly distributed 641 nodes. Grid-3: 558 nodes uniformly distributed.

The stress contour using both the exact solution and the present method is also shown in Figure 3.

4- Conclusions

In the proposed method, the homogeneous PDE is satisfied independently of its boundary conditions using the weak form of the weighted residual integral. Equilibrated Basis functions are expanded as solution bases in sub-domains called cloud, containing 98 nodes around a central node, with the approximation order equal to 4. Overlapping of the clouds ensures proper continuity of the displacement and stress throughout the domain.

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HOW TO CITE THIS ARTICLE

D. Afifi, N. Noormohammadi, B. Boroomand, Solution of 3D elasticity problems using mesh-less local equilibrated basis functions, Amirkabir J. Civil Eng., 54(11) (2023) 841-844.

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